

Turbulent Relative Particle Dispersion: Brownian Motion Theory

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Abstract

In this paper, the Brownian motion theory has been applied to the turbulent relative particle dispersion problem. The fluctuating pressure forces acting on a fluid particle are taken to follow an Uhlenbeck-Ornstein process and their correlation time appears to be very plausibly identifiable to go inversely proportional to the flow Reynolds number R_e . This connection leads to the result that the Richardson-Obukhov scaling holds only in the infinite R_e limit and disappears otherwise. Further, the Lin-Reid [11] conjecture regarding the connection between the pressure force fluctuation parameter and the energy dissipation rate in turbulence is confirmed and a R_e -dependent explicit relation between the two is determined.

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1. Introduction

Richardson [1] proposed that turbulent diffusion should be characterized by the distance between neighboring particles, because if the interparticle distance is within the inertial range, one may expect to find universal superdiffusive behavior in the relative particle dispersion process, which may be interpreted in terms of an interparticle separation dependent turbulent diffusivity. From a purely empirical analysis of atmospheric dispersion data, Richardson [1] then showed that the turbulent diffusivity defined by the rate of increase of the mean square interparticle separation distance,

$$D \equiv \frac{1}{2} \frac{d}{dt} \langle [R(t)]^2 \rangle \quad (1)$$

goes like 4/3 power of this distance,

$$D \sim \langle [R(t)]^2 \rangle^{2/3}. \quad (2)$$

Obukhov [2] showed that Richardson's relation (2) can be derived via Kolmogorov's [3] theory for homogeneous isotropic 3D fully developed turbulence (FDT). When the interparticle separation is within the inertial range, Obukhov [2] gave

$$D \sim \varepsilon^{\frac{1}{3}} \langle [R(t)]^2 \rangle^{2/3} \quad (3)$$

ε being the mean energy dissipation rate, hence the name Richardson-Obukhov (RO) scaling. On the other hand, Richardson's [1] formulation connects with the universal aspects of FDT actually stronger than Kolmogorov's formulation since the unphysical effects due to sweeping by larger scales are precluded from the onset in Richardson's [1] formulation.

The RO scaling result has received, however, little experimental support due to the difficulty of performing Lagrangian measurements over a broad enough range of time and with sufficient accuracy. Even in recent laboratory experiments (Ott and Mann [4], Sawford [5], Bourgoïn et al. [6], Salazar and Collins [7]) with high-speed photography to track particles and in numerical simulations (Yeung [8], Boffetta and Sokolov [9]) with the highest possible resolution for homogeneous isotropic turbulence, it is known to be hard to obtain an extended range with the RO scaling. Indeed, Bourgoïn et al. [6] reported that the RO scaling may not be observable in laboratory experiments even at high Reynolds numbers unless the initial particle separation is significantly small (so as to preclude the ballistic regime that currently seems to dominate the observable regime). The difficulty of achieving the RO scaling in laboratory experiments and numerical simulations appears to be due to,

- * contamination of the inertial range by dissipative effects at the ultraviolet end and by the external forcing effects at the infrared end of the spectrum caused by inadequate scale separation;
- * persistent memory of initial separation;

and is therefore basically traceable to the finiteness of the Reynolds number as well as the observational domain. The purpose of this paper is hence to shed some light on this aspect by using the Brownian motion theory applied to turbulent relative particle dispersion problem (Obukhov [10]).

2. Application of the Brownian Motion Theory

A fluid particle in a turbulent flow moves in response to fluctuations in the pressure of the surrounding fluid. The assumption of short-range correlation of the fluctuating pressure forces in a turbulent system leads to the applicability of the Brownian motion theory to the turbulent dispersion problem (Obukhov [10]). Obukhov [10] proposed that the motion of a fluid particle mimics a Markov process and is therefore described by a Fokker-Planck equation. The fluid particle is assumed to be subjected to successive small impulsive forces, like those experienced by a Brownian particle. These assumptions also hold for the relative motion of two particles in view of the negligible effect of large-scale motions in this situation (Lin and Reid [11]).

Consider the relative motion of two particles released at the same point with the same velocity in an unbounded fluid in a state of stationary homogeneous turbulence. The particles then drift apart due to different initial accelerations. We then consider the statistical average over an ensemble of such pairs. This problem is governed by the following initial-value problem,

$$\frac{dR}{dt} = V \quad (1)$$

$$\frac{dV}{dt} + \beta V = \alpha(t) \quad (2)$$

with,

$$t = 0 : R = 0, V = 0. \quad (3)$$

Here, β is a constant coefficient of resistance and $\alpha(t)$ is the fluctuating pressure force which follows a stationary, random Markov process

The solution of (1)-(3) for a particle pair is

$$V(t) = \int_0^t e^{-\beta(t-\xi)} \alpha(\xi) d\xi. \quad (4)$$

The mean square of relative velocity of an ensemble of such particle pairs is

$$\langle [V(t)]^2 \rangle = e^{-2\beta t} \int_0^t d\eta \int_0^t d\xi e^{\beta(\xi+\eta)} \langle \alpha(\xi) \alpha(\eta) \rangle. \quad (5)$$

We now assume that $\alpha(t)$ has an auto-correlation function given by (Uhlenbeck and Ornstein [12]),

$$\langle \alpha(t') \alpha(t'') \rangle = \sigma^2 e^{-\lambda(t'-t'')} \quad (6)$$

λ being the inverse of the correlation time. Large λ corresponds to white noise² (the case considered by Lin and Reid [11]) while small λ corresponds to persistent pressure fluctuation correlation.

²The white noise assumption for the fluctuating pressure forces in a real turbulent flow situation does not really seem to be justifiable.

Using (6), (5) becomes

$$\langle [V(t)]^2 \rangle = \frac{\sigma^2}{\lambda} \left[\frac{\lambda}{\beta} \left(\frac{1}{\lambda + \beta} - \frac{e^{-2\beta t}}{\lambda - \beta} \right) + \frac{2\lambda}{\lambda^2 - \beta^2} e^{-(\beta + \lambda)t} \right]. \quad (7)$$

For large λ , (7) becomes (Lin and Reid [11]),

$$\langle [V(t)]^2 \rangle \approx \frac{\sigma^2/\lambda}{\beta} (1 - e^{-2\beta t}) \quad (8)$$

which leads to

$$\langle [V(t)]^2 \rangle \approx \begin{cases} 2 \left(\frac{\sigma^2}{\lambda} \right) t, & \beta t \ll 1 \\ \frac{(\sigma^2/\lambda)}{\beta}, & \beta t \Rightarrow \infty. \end{cases} \quad (9a, b)$$

(8) or (9) shows that, in the absence of dissipation ($\beta = 0$), the particle pair experiences a runaway drifting motion due to the cumulative effect of the fluctuating pressure forces acting on the particle pair which follow a stationary process.

For small λ , (7) becomes

$$\langle [V(t)]^2 \rangle \approx \left(\frac{\sigma^2}{\lambda} \right) \frac{\lambda}{\beta^2} (1 + e^{-2\beta t} - 2e^{-\beta t}) \quad (10)$$

which leads to,

$$\langle [V(t)]^2 \rangle \approx \begin{cases} \sigma^2 t^2, & \beta t \ll 1 \\ \frac{\sigma^2}{\beta^2}, & \beta t \Rightarrow \infty. \end{cases} \quad (11a, b)$$

Comparison of (11) with (9) (see Figure 1) shows that the mean square of relative velocity initially grows more rapidly for the short-range correlation case than for the long-range correlation case.

Next, using (4), (1) and (3) give

$$R(t) = \frac{1}{\beta} \int_0^t \alpha(\xi) d\xi - \frac{1}{\beta} \int_0^t e^{-\beta(t-\xi)} \alpha(\xi) d\xi. \quad (12)$$

The relative particle dispersion is then given by

$$\begin{aligned} \langle [R(t)]^2 \rangle &= \frac{1}{\beta^2} e^{-2\beta t} \int_0^t d\eta \int_0^t d\xi e^{\beta(\xi+\eta)} \langle \alpha(\xi) \alpha(\eta) \rangle \\ &+ \frac{1}{\beta^2} \int_0^t d\eta \int_0^t d\xi \langle \alpha(\xi) \alpha(\eta) \rangle - \frac{2}{\beta^2} e^{-\beta t} \int_0^t d\eta \int_0^t d\xi e^{\beta\xi} \langle \alpha(\xi) \alpha(\eta) \rangle. \end{aligned} \quad (13)$$

Using (6), (13) becomes

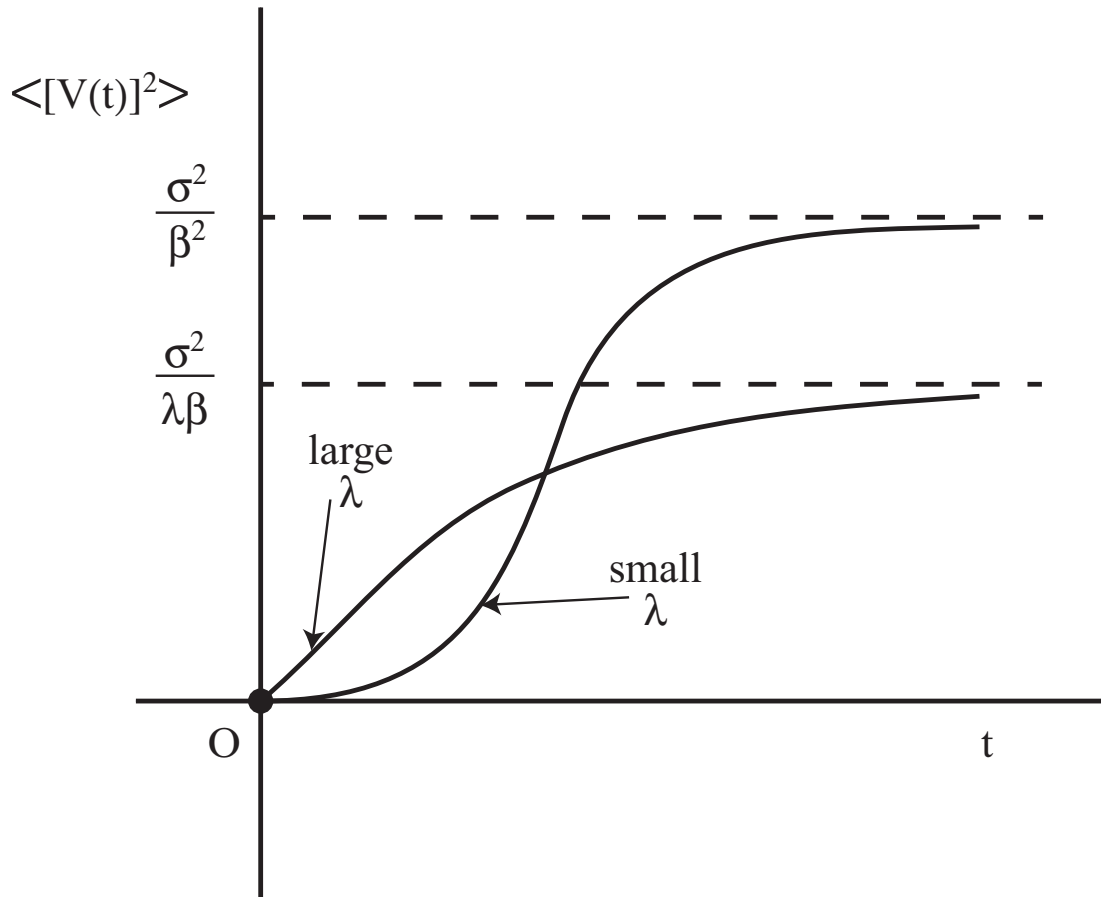


Figure 1. Effect of the fluctuating pressure force correlation time on the mean square velocity of the Brownian particle.

$$\begin{aligned}
\langle [R(t)]^2 \rangle &= \frac{(\sigma^2/\lambda)}{\beta^2} \left[\frac{\lambda}{\beta} \left(\frac{1}{\lambda + \beta} - \frac{e^{-2\beta t}}{\lambda - \beta} \right) + \frac{2\lambda}{\lambda^2 - \beta^2} e^{-(\lambda + \beta)t} \right] \\
&+ 2 \frac{(\sigma^2/\lambda)}{\beta^2} \left(t - \frac{1}{\lambda} + \frac{1}{\lambda} e^{-\lambda t} \right) - 2 \frac{(\sigma^2/\lambda)}{\beta^2} \left[\frac{2\lambda + \beta}{\beta(\lambda + \beta)} + \frac{1}{(\lambda + \beta)} e^{-(\lambda + \beta)t} \right. \\
&\left. + \frac{1}{(\lambda - \beta)} e^{-\lambda t} - \frac{2\lambda - \beta}{\beta(\lambda - \beta)} e^{-\beta t} \right].
\end{aligned} \tag{14}$$

For large λ , (14) becomes (Lin and Reid [11]),

$$\langle [R(t)]^2 \rangle \approx \frac{\sigma^2/\lambda}{\beta^3} (1 - e^{-2\beta t}) + 2 \frac{\sigma^2/\lambda}{\beta^2} t - 4 \frac{\sigma^2/\lambda}{\beta^3} (1 - e^{-\beta t}) \tag{15}$$

which leads to

$$\langle [R(t)]^2 \rangle \approx \begin{cases} \frac{2}{3} \left(\frac{\sigma^2}{\lambda} \right) t^3, & \beta t \ll 1 \\ 2 \frac{\sigma^2/\lambda}{\beta^2} t, & \beta t \Rightarrow \infty. \end{cases} \tag{16a, b}$$

(16a) represents the Richardson [1] scaling and signifies the validity of a Brownian motion theoretical framework for the relative particle dispersion problem. Indeed, if one defines a relative diffusivity D by

$$D \equiv \frac{1}{2} \frac{d}{dt} \langle [R(t)]^2 \rangle \tag{17}$$

then (16a) gives (Lin and Reid [11])

$$D = \left(\frac{\sigma^2}{\lambda} \right) t^2 = \left(\frac{3}{2} \right)^{2/3} \left(\frac{\sigma^2}{\lambda} \right)^{1/3} (\langle [R(t)]^2 \rangle)^{2/3} \tag{18}$$

which is Richardson's [1] other scaling result implying that the relative diffusivity D increases as the 4/3 power of the particle separation distance.

In view of the consistency of Richardson [1] scaling with Kolmogorov [3] scaling (which is valid in the infinite flow Reynold's number R_e limit), it therefore appears very plausible to take the Brownian parameter λ to be proportional to the flow Reynolds number R_e ,

$$\lambda = k R_e \tag{19}$$

k being a constant. It is of much interest to note that the connection between the white noise assumption for the fluctuating pressure forces and the Kolmogorov [3] large R_e assumption indicated by (19) was queried earlier by Corrsin [13].

For small λ , (14) becomes

$$\langle [R(t)]^2 \rangle \approx 2 \frac{(\sigma^2/\lambda)}{\beta^2} t + \frac{\sigma^2}{\lambda} \frac{\lambda}{\beta^4} (1 + e^{-2\beta t} - 2e^{-\beta t}) \quad (20)$$

which leads to

$$\langle [R(t)]^2 \rangle \approx \begin{cases} \frac{(\sigma^2/\lambda)}{4} \lambda t^4, & \beta t \ll 1 \\ 2 \frac{(\sigma^2/\lambda)}{\beta^2} t, & \beta t \Rightarrow \infty. \end{cases} \quad (21a, b)$$

(21a) shows that the Richardson [1] scaling holds only in the infinite R_e limit and disappears at finite R_e 's. The relative diffusivity D , defined as per (17), is now given by

$$D = \frac{1}{2} \left(\frac{\sigma^2}{\lambda} \right) \lambda t^3 = \left[4 \lambda \left(\frac{\sigma^2}{\lambda} \right) \right]^{1/4} (\langle [R(t)]^2 \rangle)^{3/4} \quad (22)$$

which shows that the relative diffusivity D , in the small R_e limit, increases as the 3/2 power of the particle separation distance.

3. Relation between the Brownian Motion Parameters and the Kolmogorov Parameters

In the Brownian motion framework, as indicated by (14), the relative particle dispersion depends on the parameter (σ^2/λ) . As pointed out by Lin and Reid [11], the parameter (σ^2/λ) looks very much like the energy dissipation rate parameter ε in the Eulerian Kolmogorov [3] framework (σ^2/λ and ε even have the same dimensions). Lin and Reid [11] therefore went on to contemplate the possibility that " $(\sigma^2/\lambda \text{ and } \varepsilon) \text{ might be identifiable, apart from a constant factor. But, in general, one can only surmise that perhaps } \frac{\sigma^2/\lambda}{\varepsilon} = f(R_e)$ ". We will now show that (19) indeed underscores such a possibility and hence contributes toward providing a Brownian motion basis for the Kolmogorov [3] theory.

Multiplying equation (2) by V and taking the average over an ensemble of such particle pairs, we obtain

$$\frac{1}{2} \frac{d}{dt} \langle [V(t)]^2 \rangle + \beta \langle [V(t)]^2 \rangle = \langle \alpha(t) V(t) \rangle. \quad (23)$$

Upon substituting (4) and using (6), we obtain

$$\frac{1}{2} \frac{d}{dt} \langle [V(t)]^2 \rangle + \beta \langle [V(t)]^2 \rangle = \frac{(\sigma^2/\lambda)}{1 + (\beta/\lambda)} [1 - e^{-(\beta+\lambda)t}]. \quad (24)$$

Further, on assuming a stationary state in the limit $t \rightarrow \infty$, (24) yields

$$\varepsilon \sim \beta \langle [V(t)]^2 \rangle = \frac{(\sigma^2/\lambda)}{1 + (\beta/\lambda)} \quad (25)$$

On using (19), (25) leads to

$$\frac{(\sigma^2/\lambda)}{\varepsilon} = f(R_e) \sim 1 + k \left(\frac{\beta}{R_e} \right) \quad (26)$$

which confirms the Lin-Reid [11] conjecture and provides one with an explicit determination of $f(R_e)$ hinted at by Lin and Reid [11].

4. Discussion

There is speculation that the difficulty in obtaining an extended range with RO scaling in both laboratory experiments and numerical simulations is due to the finiteness of the flow Reynolds number R_e in these situations. In this paper, the Brownian motion theory has been applied to the turbulent relative particle dispersion problem to shed some light on this issue. The fluctuating pressure forces acting on a fluid are taken to follow an Uhlenbeck-Ornstein process and their correlation time appears to be very plausibly inversely proportional to R_e . This confirms the connection queried earlier by Corrsin [13] between the white noise assumption for the fluctuating pressure forces and the large R_e assumption in the Kolmogorov [3] theory. This connection leads to a totally reasonable result that the RO scaling holds only in the infinite R_e limit and disappears otherwise. Further, the Lin-Reid [11] conjecture regarding the connection between the pressure force fluctuation parameter and the energy dissipation rate in FDT is confirmed and a R_e -dependent explicit relation between the two is determined.

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